

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_{11} & \mathbf{X}_{12} & \cdot & \cdot & \cdot & \mathbf{X}_{1s} \\ \mathbf{X}_{21} & \mathbf{X}_{22} & \cdot & \cdot & \cdot & \mathbf{X}_{2s} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ \mathbf{X}_{p1} & \mathbf{X}_{p2} & \cdot & \cdot & \cdot & \mathbf{X}_{ps} \end{bmatrix}$$

$$\mathbf{Z} = \begin{bmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} & \cdot & \cdot & \cdot & \mathbf{Z}_{1s} \\ \mathbf{Z}_{21} & \mathbf{Z}_{22} & \cdot & \cdot & \cdot & \mathbf{Z}_{2s} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ \mathbf{Z}_{q1} & \mathbf{Z}_{q2} & \cdot & \cdot & \cdot & \mathbf{Z}_{qs} \end{bmatrix}$$

$$\mathbf{J}_p = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \\ \vdots \\ \mathbf{1} \end{bmatrix} \quad \mathbf{diag}(\mathbf{X}\mathbf{X}') = \begin{bmatrix} \sum_{k=1}^s \mathbf{X}_{1k}^2 \\ \sum_{k=1}^s \mathbf{X}_{2k}^2 \\ \sum_{k=1}^s \mathbf{X}_{3k}^2 \\ \vdots \\ \sum_{k=1}^s \mathbf{X}_{pk}^2 \end{bmatrix} \quad \mathbf{diag}(\mathbf{Z}\mathbf{Z}') = \begin{bmatrix} \sum_{k=1}^s \mathbf{Z}_{1k}^2 \\ \sum_{k=1}^s \mathbf{Z}_{2k}^2 \\ \sum_{k=1}^s \mathbf{Z}_{3k}^2 \\ \vdots \\ \sum_{k=1}^s \mathbf{Z}_{qk}^2 \end{bmatrix}$$

The p by q matrix of squared distances between  $\mathbf{X}$  and  $\mathbf{Z}$  (individuals and stimuli) is:

$$\mathbf{D} = \begin{bmatrix} \sum_{k=1}^s (x_{1k} - z_{1k})^2 & \sum_{k=1}^s (x_{1k} - z_{2k})^2 & \cdot & \cdot & \cdot & \sum_{k=1}^s (x_{1k} - z_{qk})^2 \\ \sum_{k=1}^s (x_{2k} - z_{1k})^2 & \sum_{k=1}^s (x_{2k} - z_{2k})^2 & \cdot & \cdot & \cdot & \sum_{k=1}^s (x_{2k} - z_{qk})^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \sum_{k=1}^s (x_{pk} - z_{1k})^2 & \sum_{k=1}^s (x_{pk} - z_{2k})^2 & \cdot & \cdot & \cdot & \sum_{k=1}^s (x_{pk} - z_{qk})^2 \end{bmatrix}$$

This can be written in matrix algebra as:

$$\mathbf{D} = \mathbf{diag}(\mathbf{X}\mathbf{X}')\mathbf{J}_q - 2\mathbf{X}\mathbf{Z}' + \mathbf{J}_p\mathbf{diag}(\mathbf{Z}\mathbf{Z}') =$$

$$[\mathbf{diag}(\mathbf{X}\mathbf{X}') \mid -2\mathbf{X} \mid \mathbf{J}_p] \begin{bmatrix} \mathbf{J}'_q \\ \mathbf{Z}' \\ \mathbf{diag}(\mathbf{Z}\mathbf{Z}')' \end{bmatrix}$$

Note that the rank of  $\mathbf{D}$ ,  $\rho(\mathbf{D})$ , must be less than or equal to  $s+2$ ; i.e.,  $\rho(\mathbf{D}) \leq s + 2$

The equivalent expression for the symmetric matrix of squared distances between the stimuli is:

$$\mathbf{D}_z = \mathbf{diag}(\mathbf{Z}\mathbf{Z}')\mathbf{J}_q - 2\mathbf{Z}\mathbf{Z}' + \mathbf{J}_q\mathbf{diag}(\mathbf{Z}\mathbf{Z}') =$$

$$[\mathbf{diag}(\mathbf{Z}\mathbf{Z}') \mid -2\mathbf{Z} \mid \mathbf{J}_q] \begin{bmatrix} \mathbf{J}'_q \\ \mathbf{Z}' \\ \mathbf{diag}(\mathbf{Z}\mathbf{Z}')' \end{bmatrix}$$

### The Double-Centered Matrix -- $\mathbf{D}$

Let the mean of the  $j$ th column of  $\mathbf{D}$  be  $\mathbf{d}_{\cdot j}^2 = \frac{\sum_{i=1}^p \mathbf{d}_{ij}^2}{p} =$

$$\frac{1}{p} \sum_{i=1}^p \sum_{k=1}^s x_{ik}^2 - \frac{2}{p} \sum_{i=1}^p \sum_{k=1}^s x_{ik} z_{jk} + \sum_{k=1}^s z_{jk}^2$$

Let the mean of the  $i$ th row of  $\mathbf{D}$  be  $\mathbf{d}_i^2 = \frac{\sum_{j=1}^q \mathbf{d}_{ij}^2}{q} =$

$$\sum_{k=1}^s \mathbf{x}_{ik}^2 - \frac{2}{q} \sum_{j=1}^q \sum_{k=1}^s \mathbf{x}_{ik} \mathbf{z}_{jk} + \frac{1}{q} \sum_{j=1}^q \sum_{k=1}^s \mathbf{z}_{jk}^2$$

Let the mean of the matrix  $\mathbf{D}$  be  $\mathbf{d}_{..}^2 = \frac{\sum_{i=1}^p \sum_{j=1}^q \mathbf{d}_{ij}^2}{pq} =$

$$\frac{1}{p} \sum_{i=1}^p \sum_{k=1}^s \mathbf{x}_{ik}^2 - \frac{2}{pq} \sum_{i=1}^p \sum_{j=1}^q \sum_{k=1}^s \mathbf{x}_{ik} \mathbf{z}_{jk} + \frac{1}{q} \sum_{j=1}^q \sum_{k=1}^s \mathbf{z}_{jk}^2$$

The matrix  $\mathbf{D}$  is **Double-Centered** as follows: from each element subtract the column mean, subtract the row mean, add the matrix mean, and divide by  $-2$ ; that is,

$$\begin{aligned} y_{ij} &= \frac{(\mathbf{d}_{ij}^2 - \mathbf{d}_j^2 - \mathbf{d}_i^2 + \mathbf{d}_{..}^2)}{-2} = \sum_{k=1}^s \left[ (\mathbf{x}_{ik} - \bar{\mathbf{x}}_k)(\mathbf{z}_{jk} - \bar{\mathbf{z}}_k) \right] = \\ &= \frac{1}{-2} \left\{ \left( \sum_{k=1}^s \mathbf{x}_{ik}^2 - 2 \sum_{k=1}^s \mathbf{x}_{ik} \mathbf{z}_{jk} + \sum_{k=1}^s \mathbf{z}_{jk}^2 \right) - \left( \frac{1}{p} \sum_{i=1}^p \sum_{k=1}^s \mathbf{x}_{ik}^2 - \frac{2}{p} \sum_{i=1}^p \sum_{k=1}^s \mathbf{x}_{ik} \mathbf{z}_{jk} + \sum_{k=1}^s \mathbf{z}_{jk}^2 \right) - \right. \\ &\quad \left. \left( \sum_{k=1}^s \mathbf{x}_{ik}^2 - \frac{2}{q} \sum_{j=1}^q \sum_{k=1}^s \mathbf{x}_{ik} \mathbf{z}_{jk} + \frac{1}{q} \sum_{j=1}^q \sum_{k=1}^s \mathbf{z}_{jk}^2 \right) + \right. \\ &\quad \left. \left( \frac{1}{p} \sum_{i=1}^p \sum_{k=1}^s \mathbf{x}_{ik}^2 - \frac{2}{pq} \sum_{i=1}^p \sum_{j=1}^q \sum_{k=1}^s \mathbf{x}_{ik} \mathbf{z}_{jk} + \frac{1}{q} \sum_{j=1}^q \sum_{k=1}^s \mathbf{z}_{jk}^2 \right) \right\} = \\ &= \left\{ \sum_{k=1}^s \mathbf{x}_{ik} \mathbf{z}_{jk} - \frac{1}{p} \sum_{i=1}^p \sum_{k=1}^s \mathbf{x}_{ik} \mathbf{z}_{jk} - \frac{1}{q} \sum_{j=1}^q \sum_{k=1}^s \mathbf{x}_{ik} \mathbf{z}_{jk} + \frac{1}{pq} \sum_{i=1}^p \sum_{j=1}^q \sum_{k=1}^s \mathbf{x}_{ik} \mathbf{z}_{jk} \right\} = \\ &= \left\{ \sum_{k=1}^s \mathbf{x}_{ik} \mathbf{z}_{jk} - \sum_{k=1}^s \left( \mathbf{z}_{jk} \frac{1}{p} \sum_{i=1}^p \mathbf{x}_{ik} \right) - \sum_{k=1}^s \left( \mathbf{x}_{ik} \frac{1}{q} \sum_{j=1}^q \mathbf{z}_{jk} \right) + \sum_{k=1}^s \left( \mathbf{z}_{jk} \frac{1}{p} \sum_{i=1}^p \mathbf{x}_{ik} \right) \left( \mathbf{x}_{ik} \frac{1}{q} \sum_{j=1}^q \mathbf{z}_{jk} \right) \right\} = \\ &= \left\{ \sum_{k=1}^s \mathbf{x}_{ik} \mathbf{z}_{jk} - \sum_{k=1}^s \mathbf{z}_{jk} \bar{\mathbf{x}}_k - \sum_{k=1}^s \mathbf{x}_{ik} \bar{\mathbf{z}}_k + \sum_{k=1}^s \bar{\mathbf{z}}_k \bar{\mathbf{x}}_k \right\} = \\ &= \left\{ \sum_{k=1}^s \left[ \mathbf{z}_{jk} (\mathbf{x}_{ik} - \bar{\mathbf{x}}) \right] - \sum_{k=1}^s \left[ \bar{\mathbf{z}} (\mathbf{x}_{ik} - \bar{\mathbf{x}}) \right] \right\} = \sum_{k=1}^s \left[ (\mathbf{x}_{ik} - \bar{\mathbf{x}}_k)(\mathbf{z}_{jk} - \bar{\mathbf{z}}_k) \right] \end{aligned}$$

In matrix notation, this produces the  $p$  by  $q$  matrix  $\mathbf{Y}$  which is equal to the product of a  $p$  by  $s$  matrix  $\mathbf{X}^*$  and a  $q$  by  $s$  matrix  $\mathbf{Z}^*$ ; namely

$$\mathbf{Y} = \mathbf{X}^* \mathbf{Z}^{*'} = \begin{bmatrix} \mathbf{x}_{11} - \bar{\mathbf{x}}_1 & \mathbf{x}_{12} - \bar{\mathbf{x}}_2 & \cdot & \cdot & \cdot & \mathbf{x}_{1s} - \bar{\mathbf{x}}_s \\ \mathbf{x}_{21} - \bar{\mathbf{x}}_1 & \mathbf{x}_{22} - \bar{\mathbf{x}}_2 & \cdot & \cdot & \cdot & \mathbf{x}_{2s} - \bar{\mathbf{x}}_s \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mathbf{x}_{p1} - \bar{\mathbf{x}}_1 & \mathbf{x}_{p2} - \bar{\mathbf{x}}_2 & \cdot & \cdot & \cdot & \mathbf{x}_{ps} - \bar{\mathbf{x}}_s \end{bmatrix} \begin{bmatrix} \mathbf{z}_{11} - \bar{\mathbf{z}}_1 & \mathbf{z}_{12} - \bar{\mathbf{z}}_2 & \cdot & \cdot & \cdot & \mathbf{z}_{1s} - \bar{\mathbf{z}}_s \\ \mathbf{z}_{21} - \bar{\mathbf{z}}_1 & \mathbf{z}_{22} - \bar{\mathbf{z}}_2 & \cdot & \cdot & \cdot & \mathbf{z}_{2s} - \bar{\mathbf{z}}_s \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mathbf{z}_{q1} - \bar{\mathbf{z}}_1 & \mathbf{z}_{q2} - \bar{\mathbf{z}}_2 & \cdot & \cdot & \cdot & \mathbf{z}_{qs} - \bar{\mathbf{z}}_s \end{bmatrix}'$$

Note the reason why this is called a *double* centered matrix. The  $\mathbf{x}_i$ 's are in a coordinate system *centered at their mean* --  $\bar{\mathbf{x}}$  -- and the  $\mathbf{z}_j$  's are in *an entirely different coordinate system* -- one centered at their mean --  $\bar{\mathbf{z}}$ .

### The Double-Centered Matrix -- $\mathbf{D}_z$

$$\text{Let the mean of the } j\text{th column of } \mathbf{D}_z \text{ be } \mathbf{d}_{\cdot j}^2 = \frac{\sum_{m=1}^q \mathbf{d}_{mj}^2}{q}$$

$$\text{Let the mean of the } m\text{th row of } \mathbf{D}_z \text{ be } \mathbf{d}_{m \cdot}^2 = \frac{\sum_{j=1}^q \mathbf{d}_{mj}^2}{q}$$

$$\text{Let the mean of the matrix } \mathbf{D}_z \text{ be } \mathbf{d}_{\cdot\cdot}^2 = \frac{\sum_{m=1}^q \sum_{j=1}^q \mathbf{d}_{mj}^2}{q^2}$$

The matrix  $\mathbf{D}_z$  is **Double-Centered** as follows: from each element subtract the column mean, subtract the row mean, add the matrix mean, and divide by  $-2$ ; that is,

$$y_{mj} = \frac{(\mathbf{d}_{mj}^2 - \mathbf{d}_{\cdot j}^2 - \mathbf{d}_{m \cdot}^2 + \mathbf{d}_{\cdot\cdot}^2)}{-2} = \sum_{k=1}^s (\mathbf{z}_{mk} - \bar{\mathbf{z}}_k)(\mathbf{z}_{jk} - \bar{\mathbf{z}}_k)$$

In matrix notation, this produces the  $q$  by  $q$  matrix  $\mathbf{Y}$  which is equal to the cross-product matrix of the  $q$  by  $s$  matrix  $\mathbf{Z}^*$  times itself; namely

$$Y = Z^* Z^{*'} = \begin{bmatrix} z_{11} - \bar{z}_1 & z_{12} - \bar{z}_2 & \cdot & \cdot & \cdot & z_{1s} - \bar{z}_s \\ z_{21} - \bar{z}_1 & z_{22} - \bar{z}_2 & \cdot & \cdot & \cdot & z_{2s} - \bar{z}_s \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ z_{q1} - \bar{z}_1 & z_{q2} - \bar{z}_2 & \cdot & \cdot & \cdot & z_{qs} - \bar{z}_s \end{bmatrix} \begin{bmatrix} z_{11} - \bar{z}_1 & z_{12} - \bar{z}_2 & \cdot & \cdot & \cdot & z_{1s} - \bar{z}_s \\ z_{21} - \bar{z}_1 & z_{22} - \bar{z}_2 & \cdot & \cdot & \cdot & z_{2s} - \bar{z}_s \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ z_{q1} - \bar{z}_1 & z_{q2} - \bar{z}_2 & \cdot & \cdot & \cdot & z_{qs} - \bar{z}_s \end{bmatrix}'$$

Note that this problem is easily solved because, without loss of generality, we can assume that the coordinates have zero means, that is,  $\bar{z} = \mathbf{0}$ , and we can use simple eigenvector/eigenvalue decomposition to solve for  $Z$ .

### Example From Old Homework

Below is a matrix of squared distances between 7 points in two dimensions. One of the points is at the origin and the other 6 are arranged symmetrically around it.

Double Center the matrix and solve for the coordinates. Show all your computations.

	1	2	3	4	5	6	7
1	0	2	1	2	2	1	2
2	2	0	1	4	8	5	4
3	1	1	0	1	5	4	5
4	2	4	1	0	4	5	8
5	2	8	5	4	0	1	4
6	1	5	4	5	1	0	1
7	2	4	5	8	4	1	0

Find the row, column, and matrix means:

Row & Column 1 =  $(0+2+1+2+2+1+2)/7 = 10/7 = 1.43$   
 Row & Column 2 =  $(2+0+1+4+8+5+4)/7 = 24/7 = 3.43$   
 Row & Column 3 =  $(1+1+0+1+5+4+5)/7 = 17/7 = 2.43$   
 Row & Column 4 =  $(2+4+1+0+4+5+8)/7 = 24/7 = 3.43$   
 Row & Column 5 =  $(2+8+5+4+0+1+4)/7 = 24/7 = 3.43$   
 Row & Column 6 =  $(1+5+4+5+1+0+1)/7 = 17/7 = 2.43$   
 Row & Column 7 =  $(2+4+5+8+4+1+0)/7 = 24/7 = 3.43$

Matrix Mean =

$(0+2+1+2+2+1+2+2+0+1+4+8+5+4+1+1+0+1+5+4+5+2+4+1+0+4+5+8+2+8+5+4+0+1+4+1+5+4+5+1+0+1+2+4+5+8+4+1+0)/49 = 140/49 = 2.86$

This produces the Double-Centered Matrix:

	1	2	3	4	5	6	7
1	0	0	0	0	0	0	0
2	0	2	1	0	-2	-1	0
3	0	1	1	1	-1	-1	-1
4	0	0	1	2	0	-1	-2
5	0	-2	-1	0	2	1	0
6	0	-1	-1	-1	1	1	1
7	0	0	-1	-2	0	1	2

Answer???

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \\ -1 & 1 \\ -1 & -1 \\ 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 & -1 & -1 & -1 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & -2 & -1 & 0 \\ 0 & 1 & 1 & 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 2 & 0 & -1 & -2 \\ 0 & -2 & -1 & 0 & 2 & 1 & 0 \\ 0 & -1 & -1 & -1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -2 & 0 & 1 & 2 \end{bmatrix}$$

So that  $\begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \\ -1 & 1 \\ -1 & -1 \\ 0 & -1 \\ 1 & -1 \end{bmatrix}$  is an answer.

### Remark About Rotations

Any *rotation* of above matrix is also an answer. For example,

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \\ -1 & 1 \\ -1 & -1 \\ 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} .8 & -.6 \\ .6 & .8 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1.4 & .2 \\ .6 & .8 \\ -.2 & 1.4 \\ -1.4 & -.2 \\ -.6 & -.8 \\ .2 & -1.4 \end{bmatrix}$$

In general:

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \\ -1 & 1 \\ -1 & -1 \\ 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

Note that:

$$\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In three dimensions rotation matrices look like this:

$$\Gamma_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_x & -\sin \theta_x \\ 0 & \sin \theta_x & \cos \theta_x \end{bmatrix}, \text{ a rotation around the x-axis;}$$

$$\Gamma_y = \begin{bmatrix} \cos \theta_y & 0 & \sin \theta_y \\ 0 & 1 & 0 \\ -\sin \theta_y & 0 & \cos \theta_y \end{bmatrix}, \text{ a rotation around the y-axis;}$$

$$\Gamma_z = \begin{bmatrix} \cos \theta_z & -\sin \theta_z & 0 \\ \sin \theta_z & \cos \theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ a rotation around the z-axis.}$$

Note that

$$\Gamma_x \Gamma_x' = \mathbf{I}_3$$

$$\Gamma_y \Gamma_y' = \mathbf{I}_3$$

$$\Gamma_z \Gamma_z' = \mathbf{I}_3$$

Where  $\mathbf{I}_3$  is a 3 by 3 Identity matrix.

You can get any general rotation by simply multiplying the above matrices; for example:

$$\Gamma_x \Gamma_y \Gamma_z$$

$$\Gamma_x \Gamma_z$$

$$\Gamma_x \Gamma_z \Gamma_y$$

**Etc**